CHAPTER 8

Differentiation under the Integral Sign. Improper Integrals. The Gamma Function

1. Differentiation under the Integral Sign

We recall the elementary integration formula

$$\int_0^1 t^n\,dt = \left[\frac{1}{n+1}t^{n+1}\right]_0^1 = \frac{1}{n+1},$$

valid for any $n > -1$. Since n need not be an integer, we employ the variable x and write

$$\phi(x) = \int_0^1 t^x\,dt = \frac{1}{x+1}, \qquad x > -1. \tag{1}$$

Suppose we wish to compute the derivative $\phi'(x)$. We can proceed in two ways. Equating the first and last expressions in (1), we have

$$\phi(x) = \frac{1}{x+1}, \qquad \phi'(x) = -\frac{1}{(x+1)^2}.$$

On the other hand, we may try the following procedure:

$$\frac{d}{dx}\phi(x) = \frac{d}{dx}\int_0^1 t^x\,dt = \int_0^1 \frac{d}{dx}(t^x)\,dt = \int_0^1 t^x \log t\,dt. \tag{2}$$

Is it true that

$$\int_0^1 t^x \log t\,dt = -\frac{1}{(x+1)^2}, \tag{3}$$

at least for $x > -1$? In this section we shall determine conditions under which a process such as (2) is valid. To examine the validity of differentiation under the integral sign, as the process (2) is called, we first develop a property of continuous functions on R^2.

Let S be a region in R^2 and $f: S \to R^1$ a continuous function. We recall that f is continuous at a point $(x_0, y_0) \in S$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f(x, y) - f(x_0, y_0)| < \varepsilon$$

whenever

$$|x - x_0| + |y - y_0| < \delta.$$

It is important to note that the size of δ depends not only on the size of ε but also on the particular point (x_0, y_0) at which continuity is defined. If the size of δ depends only on ε and not on the point (x_0, y_0), then f is said to be *uniformly continuous on S*. That is, f is *uniformly continuous* if for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|f(x', y') - f(x'', y'')| < \varepsilon$$

for **all** (x', y'), (x'', y'') in S which satisfy the inequality

$$|x' - x''| + |y' - y''| < \delta.$$

In other words, the size of δ depends only on ε.

We denote the boundary of a region S in R^2 by ∂S. A region in R^2 is said to be **bounded** if it is contained in a sufficiently large disk. A region S is **closed** if it contains its boundary, ∂S. The basic theorem concerning uniformly continuous function states that *a function f which is continuous on a closed bounded region is uniformly continuous*. The same result holds in any number of dimensions. We omit the proof.

Suppose a function ϕ is given by the formula

$$\phi(x) = \int_c^d f(x, t)\,dt, \qquad a \le x \le b,$$

where c and d are constants. If the integration can be performed explicitly, then $\phi'(x)$ can be found by a computation. However, even when the evaluation of the integral is impossible, it sometimes happens that $\phi'(x)$ can be found. The basic formula is given in the next theorem, known as **Leibniz' Rule**.

Theorem 1. *Suppose that ϕ is defined by*

$$\phi(x) = \int_c^d f(x, t)\,dt, \qquad a \le x \le b, \tag{4}$$

where c and d are constants. If f and f_x are continuous in the rectangle

$$R = \{(x, t) : a \le x \le b, \quad c \le t \le d\},$$

then

$$\phi'(x) = \int_c^d f_x(x, t)\,dt, \qquad a < x < b. \tag{5}$$

That is, the derivative may be found by differentiating under the integral sign.

PROOF. We prove the theorem by showing that the difference quotient

$$[\phi(x+k) - \phi(x)]/k$$

tends to the right side of (5) as k tends to zero. If x is in (a, b) then, from (4), we have

$$\begin{aligned}\frac{\phi(x+k)-\phi(x)}{k} &= \frac{1}{k}\int_c^d f(x+k,t)\,dt - \frac{1}{k}\int_c^d f(x,t)\,dt \\ &= \frac{1}{k}\int_c^d [f(x+k,t) - f(x,t)]\,dt.\end{aligned}$$

Since differentiation and integration are inverse processes, we can write

$$f(x+k,t) - f(x,t) = \int_x^{x+k} f_\xi(\xi,t)\,d\xi,$$

and so

$$\frac{\phi(x+k)-\phi(x)}{k} = \frac{1}{k}\int_c^d \int_x^{x+k} f_\xi(\xi,t)\,d\xi\,dt.$$

We note that f_x is uniformly continuous on R, since a function which is continuous on a bounded, closed set is uniformly continuous there. Therefore, using the comma notation for the derivative with respect to the first variable, if $\varepsilon > 0$ is given, there is a $\delta > 0$ such that

$$|f_{,1}(\xi,t) - f_{,1}(x,t)| < \frac{\varepsilon}{d-c}$$

for all t in $[c, d]$ and all ξ with $|\xi - x| < \delta$. We now wish to show that

$$\frac{\phi(x+k)-\phi(x)}{k} - \int_c^d f_{,1}(x,t)\,dt \to 0 \qquad \text{as } k \to 0.$$

We write

$$\int_c^d f_{,1}(x,t)\,dt = \frac{1}{k}\int_c^d \int_x^{x+k} f_{,1}(x,t)\,d\xi\,dt,$$

which is true because the integrand on the right does not contain ξ. Substituting this last expression in the one above, we find, for $0 < |k| < \delta$,

$$\begin{aligned}&\left|\frac{\phi(x+k)-\phi(x)}{k} - \int_c^d f_{,1}(x,t)\,dt\right| \\ &= \left|\int_c^d \left\{\frac{1}{k}\int_x^{x+k} [f_{,1}(\xi,t) - f_{,1}(x,t)]\,d\xi\right\} dt\right| \\ &\le \int_c^d \left|\frac{1}{k}\int_x^{x+k} \frac{\varepsilon}{d-c}\,d\xi\right| dt = \frac{\varepsilon}{(d-c)}\cdot(d-c) = \varepsilon.\end{aligned}$$

Since ε is arbitrary, the theorem follows.

Theorem 1 shows that the formula (3) is justified for $x > 0$, since the integrand $f(x, t)$ is then continuous in an appropriate rectangle. Later we shall examine more closely the validity of (3) when $-1 < x \le 0$, in which case the integral is improper.

EXAMPLE 1. Find the value of $\phi'(x)$ if

$$\phi(x) = \int_0^{\pi/2} f(x, t)\,dt; \qquad f(x, t) = \begin{cases} \dfrac{\sin xt}{t} & \text{if } t \neq 0, \\ x & \text{if } t = 0. \end{cases}$$

SOLUTION. Since

$$\lim_{t \to 0} \frac{\sin xt}{t} = x \lim_{t \to 0} \frac{\sin xt}{xt} = x,$$

the integrand is continuous for $0 \le t \le \pi/2$ and for all x. Also, we have

$$f_x(x, t) = \begin{cases} \cos xt & \text{if } t \neq 0, \\ 1 = \cos xt & \text{if } t = 0, \end{cases}$$

so $f_x(x, t)$ is continuous everywhere. Therefore

$$\phi'(x) = \int_0^{\pi/2} \cos xt\,dt = -\left[\frac{1}{x}\sin xt\right]_0^{\pi/2} = -\frac{\sin(\pi/2)x}{x}, \qquad x \neq 0.$$

It is a fact that the integral expression for ϕ cannot be evaluated explicitly.

EXAMPLE 2. Evaluate

$$\int_0^1 \frac{du}{(u^2 + 1)^2}$$

by letting

$$\phi(x) = \int_0^1 \frac{du}{u^2 + x} = \frac{1}{\sqrt{x}} \arctan(1/\sqrt{x})$$

and computing $-\phi'(1)$.

SOLUTION.

$$\phi'(x) = -\int_0^1 \frac{du}{(u^2 + x)^2} = \frac{1}{\sqrt{x}} \frac{-\frac{1}{2}x^{-3/2}}{1 + (1/x)} - \frac{1}{2x\sqrt{x}} \arctan\frac{1}{\sqrt{x}}$$

and

$$-\phi'(1) = \int_0^1 \frac{du}{(u^2 + 1)^2} = \frac{1}{2}\left(\frac{1}{2} + \arctan 1\right) = \frac{1}{2}\left(\frac{1}{2} + \frac{\pi}{4}\right).$$

Leibniz' Rule may be extended to the case where the limits of integration

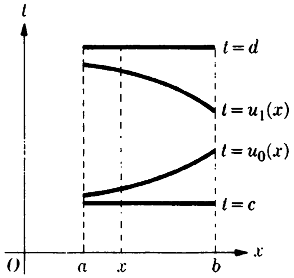

Fig. 8-1

also depend on x. We consider a function defined by

$$\phi(x) = \int_{u_0(x)}^{u_1(x)} f(x,t)\,dt, \tag{6}$$

where $u_0(x)$ and $u_1(x)$ are continuously differentiable functions for $a \leq x \leq b$. Furthermore, the ranges of u_0 and u_1 are assumed to lie between c and d (Fig. 8-1).

To obtain a formula for the derivative $\phi'(x)$, where ϕ is given by an integral such as (6), it is simpler to consider a new integral which is more general than (6). We define

$$F(x,y,z) = \int_y^z f(x,t)\,dt \tag{7}$$

and obtain the following corollary of Leibniz' Rule.

Theorem 2. *Suppose that f satisfies the conditions of Theorem* 1 *and that F is defined by* (7) *with $c < y, z < d$. Then*

$$\frac{\partial F}{\partial x} = \int_y^z f_{,1}(x,t)\,dt, \tag{8a}$$

$$\frac{\partial F}{\partial y} = -f(x,y), \tag{8b}$$

$$\frac{\partial F}{\partial z} = f(x,z). \tag{8c}$$

Proof. Formula (8a) is Theorem 1. Formulas (8b) and (8c) are precisely the Fundamental Theorem of Calculus, since taking the partial derivative of F with respect to one variable, say y, implies that x and z are kept *fixed.*

Theorem 3 (General Rule for Differentiation under the Integral Sign). *Suppose that f and $\partial f/\partial x$ are continuous in the rectangle*

$$R = \{(x,t) : a \leq x \leq b, \quad c \leq t \leq d\},$$

and suppose that $u_0(x)$, $u_1(x)$ are continuously differentiable for $a \leq x \leq b$

with the range of u_0 and u_1 in (c, d). If ϕ is given by

$$\phi(x) = \int_{u_0(x)}^{u_1(x)} f(x, t)\,dt,$$

then

$$\phi'(x) = f[x, u_1(x)]u_1'(x) - f[x, u_0(x)] \cdot u_0'(x) + \int_{u_0(x)}^{u_1(x)} f_x(x, t)\,dt. \tag{9}$$

PROOF. We observe that

$$F(x, u_0(x), u_1(x)) = \phi(x)$$

in Theorem 2. Applying the Chain Rule, we get

$$\phi'(x) = F_x + F_y u_0'(x) + F_z u_1'(x).$$

Inserting the values of F_x, F_y, and F_z from (8), we obtain the desired result (9).

EXAMPLE 3. Find $\phi'(x)$, given that

$$\phi(x) = \int_0^{x^2} \arctan\frac{t}{x^2}\,dt.$$

SOLUTION. We have

$$\frac{\partial}{\partial x}\left(\arctan\frac{t}{x^2}\right) = \frac{-2t/x^3}{1 + (t^2/x^4)} = -\frac{2tx}{t^2 + x^4}.$$

We use formula (9) and find

$$\phi'(x) = (\arctan 1) \cdot (2x) - \int_0^{x^2} \frac{2tx\,dt}{t^2 + x^4}.$$

Setting $t = x^2 u$ in the integral on the right, we obtain

$$\phi'(x) = \frac{\pi x}{2} - \int_0^1 \frac{2x^3 u \cdot x^2\,du}{x^4 u^2 + x^4} = \frac{\pi x}{2} - x\int_0^1 \frac{2u\,du}{u^2 + 1} = x\left(\frac{\pi}{2} - \log 2\right).$$

PROBLEMS

In each of Problems 1 through 5, express $\phi'(x)$ as a definite integral, using Leibniz' Rule.

1. $\phi(x) = \int_0^1 \frac{\sin xt\,dt}{1 + t}$
2. $\phi(x) = \int_0^2 \frac{e^{-xt}\,dt}{1 + t^2}$
3. $\phi(x) = \int_1^2 \frac{e^{-t}\,dt}{1 + xt}$
4. $\phi(x) = \int_0^1 \frac{t^2\,dt}{(1 + xt)^2}$